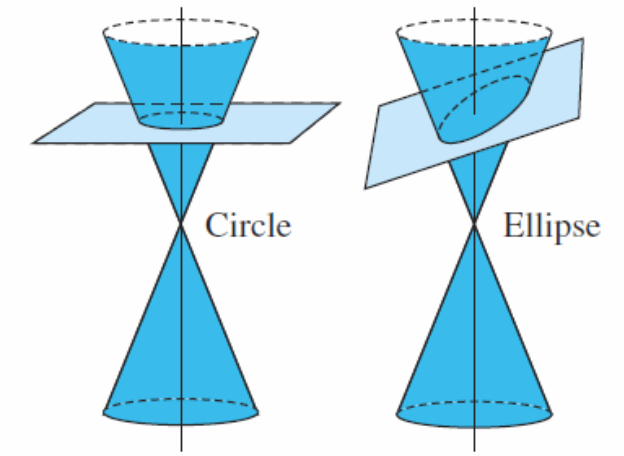
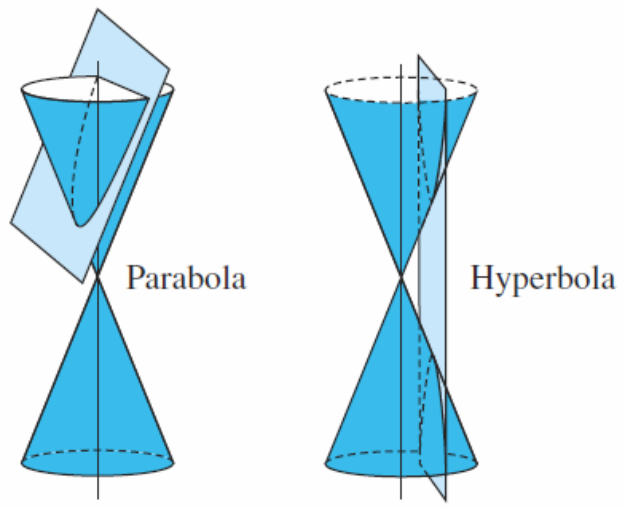
Chapter 7 – Conic Sections Overview





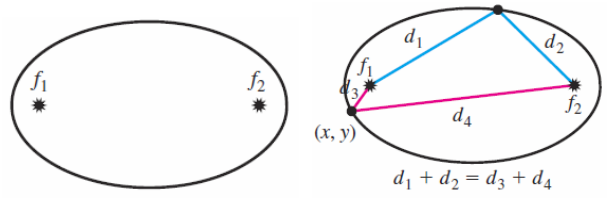
Circle Review....

Definition: A **<u>circle</u>** is the set of points, P, in a plane which are a fixed distance (radius) from a fixed point (center).

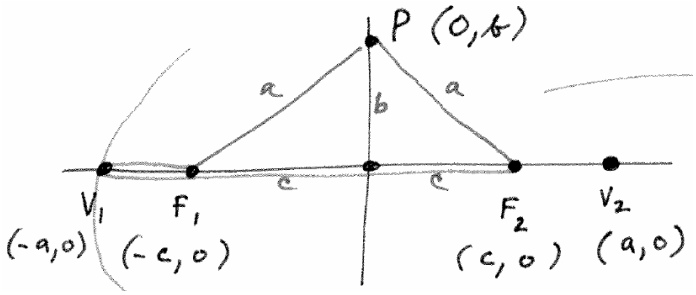
$$(x-h)^2+(y-k)^2=r^2$$

7.2 – The Ellipse

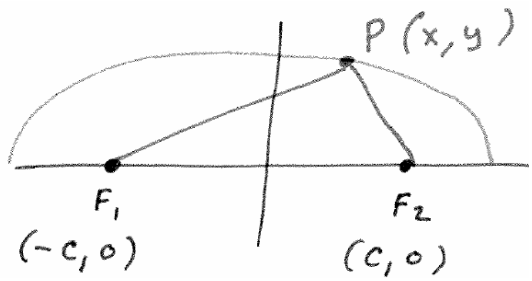
Definition: An **<u>ellipse</u>** is the set of points, P, in a plane such that the _____ of the distances from P to two fixed points is constant.



<u>Derrivation Case #1</u>: P is at "high noon"



Derrivation Case #2: P is arbitrary



From our definition, $dist(F_1P) + dist(F_2P)$ *is constant*

$$\sqrt{(x+c)^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$$

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$

$$(x+c)^2+y^2=\left(2a-\sqrt{(x-c)^2+y^2}\right)^2$$

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

$$(x+c)^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2$$

$$x^2+2xc+c^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2xc+c^2$$

$$2xc=4a^2-4a\sqrt{(x-c)^2+y^2}-2xc$$

$$4xc=4a^2-4a\sqrt{(x-c)^2+y^2}$$

$$4xc=4a^2-4a\sqrt{x^2-2xc+c^2+y^2}$$

$$a^2-xc=a\sqrt{x^2-2xc+c^2+y^2}$$

$$a^4-2a^2xc+x^2c^2=a^2(x^2-2xc+c^2+y^2)$$

$$a^4-2a^2xc+x^2c^2=a^2x^2-2a^2xc+a^2c^2+a^2y^2$$

$$a^4+x^2c^2=a^2x^2+a^2c^2+a^2y^2$$

Substitute in $c^2=a^2-b^2$

$$a^4+x^2(a^2-b^2)=a^2x^2+a^2(a^2-b^2)+a^2y^2$$

$$a^4+x^2a^2-x^2b^2=a^2x^2+a^4-a^2b^2+a^2y^2$$

$$-x^2b^2=-a^2b^2+a^2y^2$$

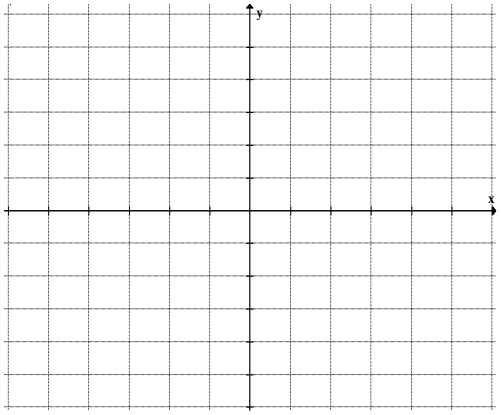
$$-x^2b^2-a^2y^2=-a^2b^2$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

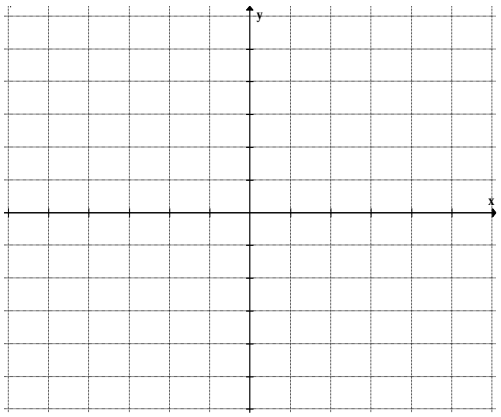
General Formula for an ellipse, centered at (h,k): $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

To find the foci, use _____.

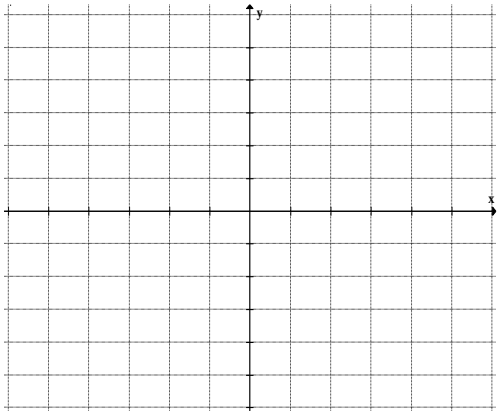
1. Graph. Label the foci. $\frac{(x+2)^2}{16}+\frac{(y-1)^2}{4}$



2. Graph. Label the foci. $\frac{(x-3)^2}{1}+\frac{(y-2)^2}{4}=1$



3. Graph. Label the foci. $\frac{(x+1)^2}{4} + \frac{(y-2)^2}{4} = 1$



4. Find the center and foci:

$x^2 + 4y^2 - 8y + 4x - 8 = 0$

5. Find the center and foci:

$6x^2 + 3y^2 - 24x + 18y - 3 = 0$

6. What happens if you get fractions?

$4(x-3)^2 + 7(y+2)^2 = 10$

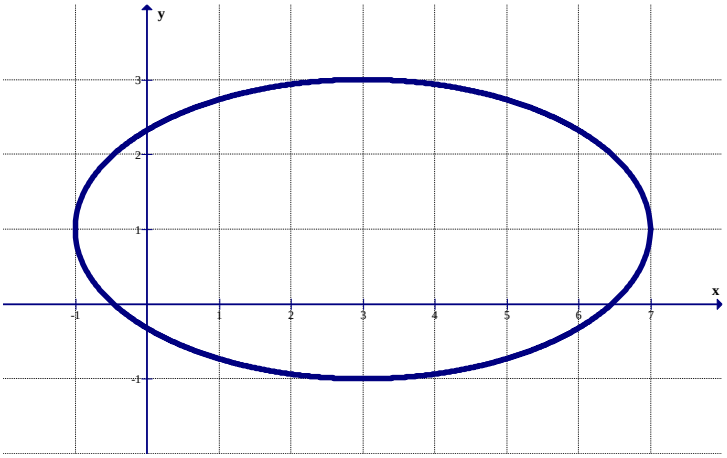
7. Find the equation of the ellipse having the following characteristics:

Vertices at $(\pm 6,0)$ and foci at $(\pm 4,0)$.

8. Find the equation of the ellipse having the following characteristics:

Foci at $(-4,-3)$ and $(8,-3)$; length of the minor axis is 8 units

9. Find the equation of the ellipse from its graph:



7.3 – The Hyperbola

Definition: A **hyperbola** is the set of points, P, in a plane such that the _____ of the distances from P to two fixed points is constant.

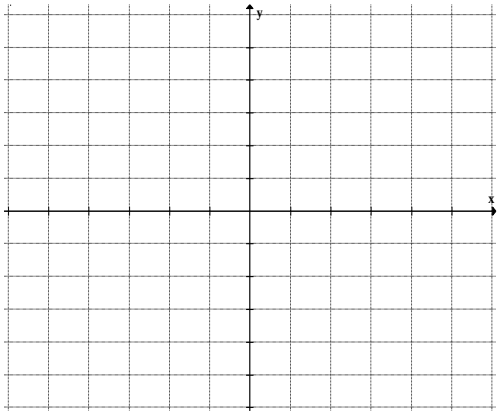
General Formulas for a hyperbola, centered at (h, k):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

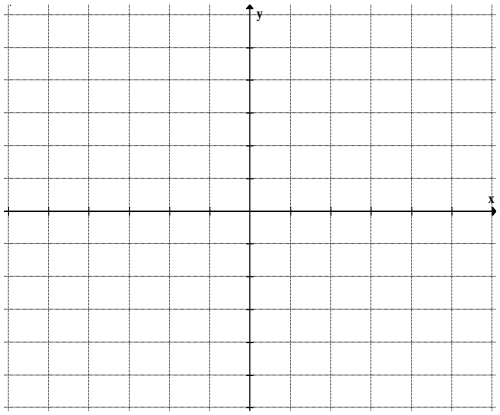
To find the foci, use _____.

1. Graph and analyze: $\frac{(y-2)^2}{4} - \frac{(x+1)^2}{16} = 1$



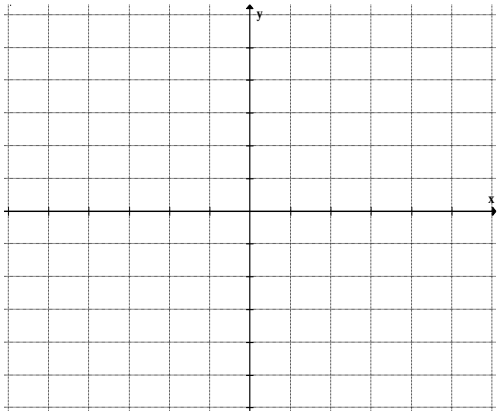
2. Graph and analyze:

$$4x^2 - 9y^2 - 24x + 72y - 144 = 0$$



3. Graph and analyze:

$$10x^2 - 5y^2 + 60x + 20y - 20 = 0$$



4. Find the equation of the hyperbola, given the following characteristics:

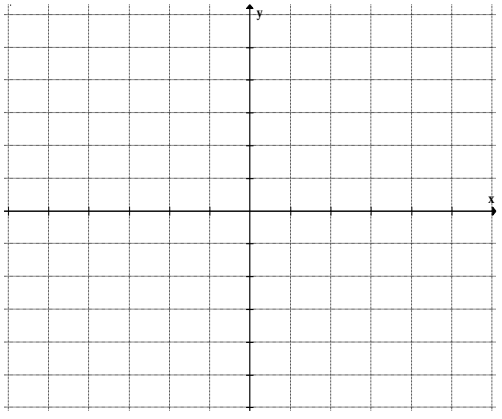
Vertices at $(\pm 6,0)$ and foci at $(\pm 8,0)$.

5. Find the equation of the hyperbola, given the following characteristics:

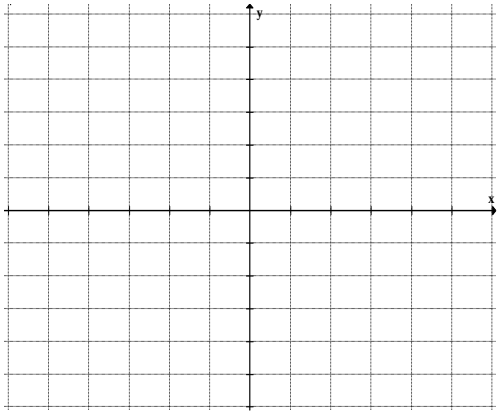
Foci at $(-5,2)$ and $(7,2)$; lengh of the conjugate axis is 8 units.

6. Graph. Find the foci and the asymptotes.

$$4x^2 - 36y^2 - 40x + 148y - 188 = 0$$



7. Graph. Find the foci. $\frac{(x+2)^2}{9} + \frac{(y-1)^2}{4} = 1$



7.4 – The Parabola

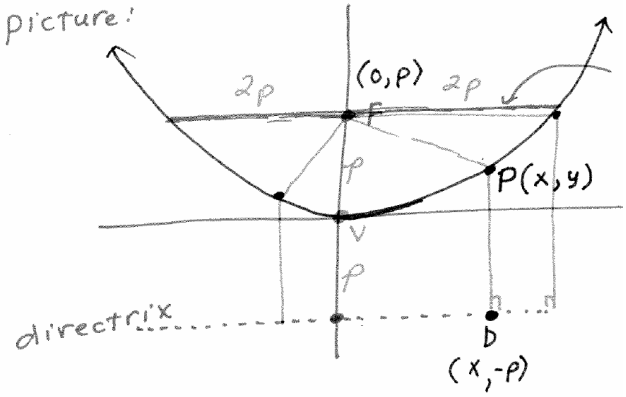
Recall the "old way" of graphing a parabola:
$y = a(x-h)^2 + k$

vertex:

max/min value:

axis of symmetry:

Definition: A **parabola** is the set of points, P, in a plane which are _____ from a fixed point (focus) and a fixed line (called the _____).



From the definition, $dist(FP) = dist(PD)$

$$\sqrt{(x-0)^2 + (y-p)^2} = \sqrt{(x-x)^2 + (y+p)^2}$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

$$x^2 = 4py$$

General Formulas:

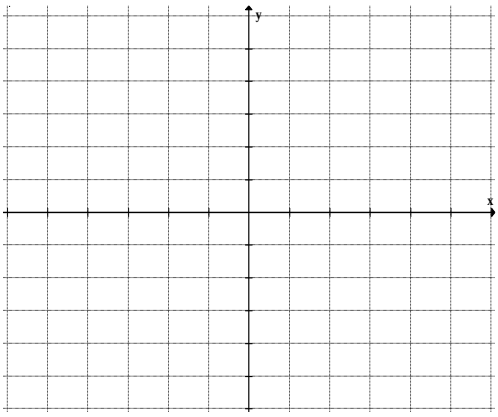
$(x-h)^2 = 4p(y-k)$

$(y-k)^2 = 4p(x-h)$

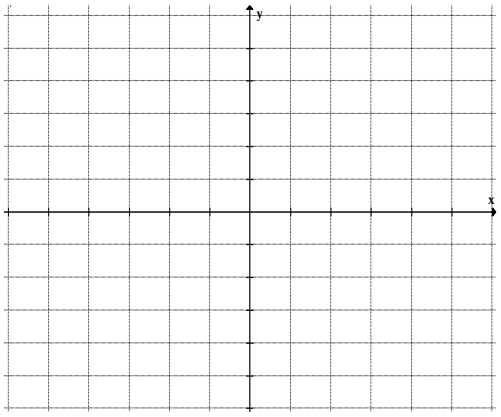
Connection between the "old way" and the "conic way"...

*For each parabola, identify the vertex, focus, directrix, axis of symmetry, and max/min value (if applicable). Graph each.

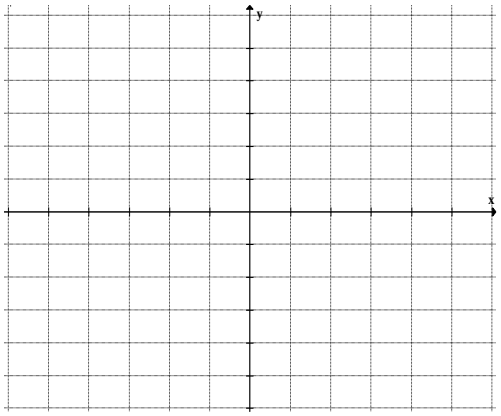
1. $(y+2)^2 = -4(x-1)$



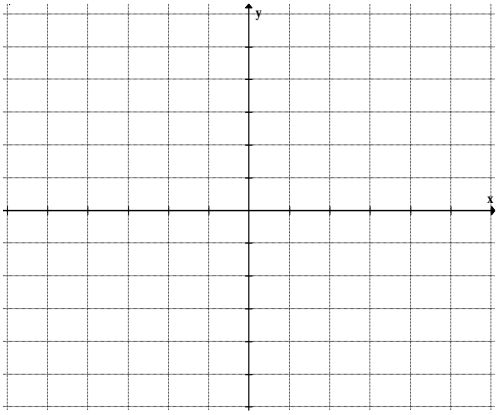
2. $(x+1)^2 = 8(y+3)$



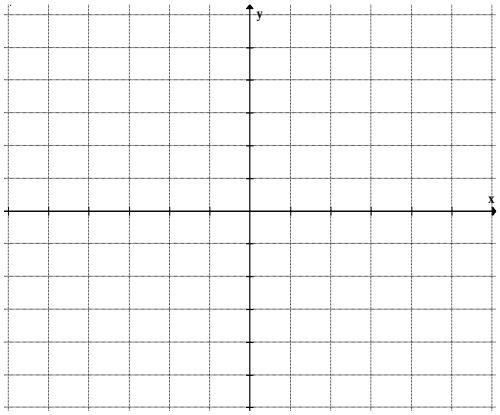
3. $(x-2)^2 = -6(y+1)$



4. $x^2 - 10x - 12y + 25 = 0$



5. $y^2 - 2y + 8x + 9 = 0$



6. $3y^2 + 10y + 7x - 13 = 0$

7. Find the equation of the parabola with focus at $(1,2)$ and directrix $y = 5$.

8. Find the equation of the parabola with focus at $(-3,4)$ and directrix $x = -4$.

9. Find the focus and directrix:

$$y = \frac{-3}{2}(x+2)^2 - 4$$

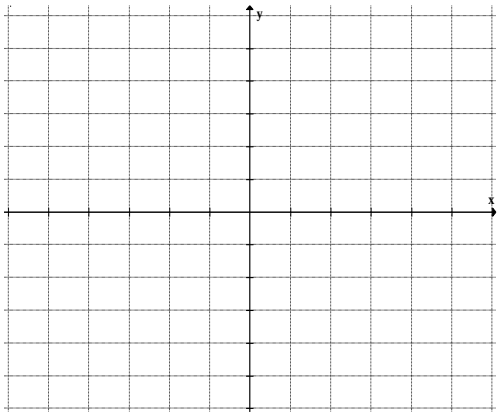
10. Find the focus and directrix:

$$x = \frac{1}{8}(y-3)^2 + 2$$

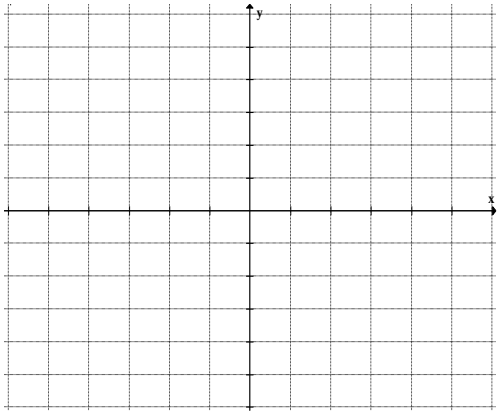
SHAPE	Formulas	Comments
LINE	$ax+by=c$ standard $y=mx+b$ slope-intercept $y-y_1=m(x-x_1)$ point-slope	no exponents $m=\frac{y_2-y_1}{x_2-x_1}$
Parabola	$y=a(x-h)^2+k$ $(x-h)^2=4p(y-k)$ ∪ $(y-k)^2=4p(x-h)$ ⊂	Either x or y is squared. * p is the distance from the vertex to the < focus / directrix $a=\frac{1}{4p}$ $p=\frac{1}{4a}$
Circle	$(x-h)^2+(y-k)^2=r^2$	x^2+y^2 (Same coefficient)
Ellipse	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	x^2+y^2 (diff. coefficient) Bigger denom = major axis foci are on major axis $c^2=a^2-b^2$ c is dist. from center to focus.
Hyperbola	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ ⊃⊂ $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ ∪ ∩ Asymptotes: $y-k=\pm m(x-h)$	If x^2 first, opens L/R. If y^2 first, opens up/down $c^2=a^2+b^2$ c is dist. from center to focus a is dist from center to vertex
Exp	$y=a^x$ 	d: $x\in\mathbb{R}$ passes through r: $y>0$ (0,1) and (1,a)
Log	$y=\log_a x$ 	d: $x>0$ passes through r: $y\in\mathbb{R}$ (1,0) and (a,1)

Chapter 7 Review

1. Graph and analyze. $\frac{(x-2)^2}{16} - \frac{(y+1)^2}{4} = 1$

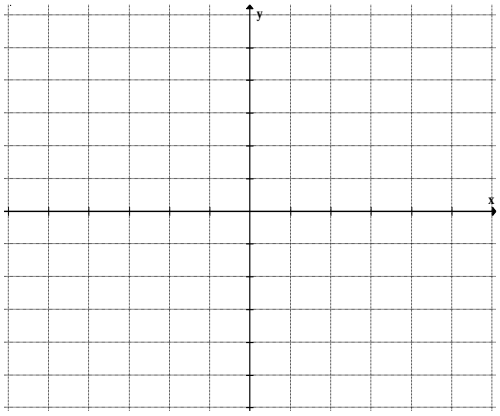


2. Graph and analyze. $\frac{(x+1)^2}{4} + \frac{(y+2)^2}{25} = 1$



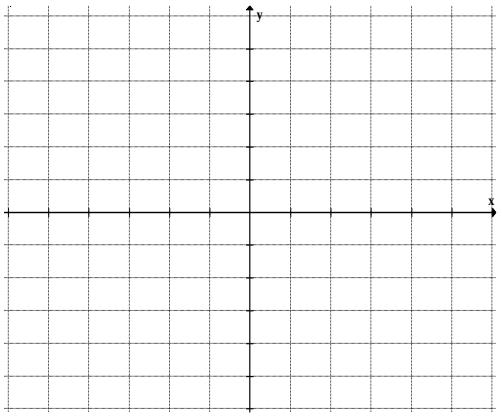
3. Find the vertices and foci. Graph.

$$2x^2 + y^2 + 8x - 6y + 1 = 0$$



4. Find the vertices, foci and asymptotes. Graph.

$$4x^2 - 9y^2 - 8x + 18y + 31 = 0$$

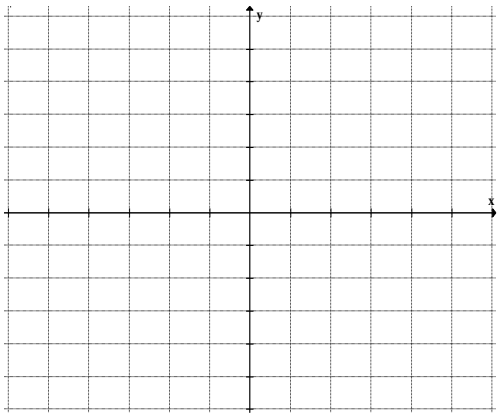


5. Find the equation of the _____ having vertices at $(-3,2)$ and $(9,2)$ and foci at $(-5,2)$ and $(11,2)$.

6. Find the equation of the ellipse having a major axis of length 10 and foci at $(5,2)$ and $(1,2)$.

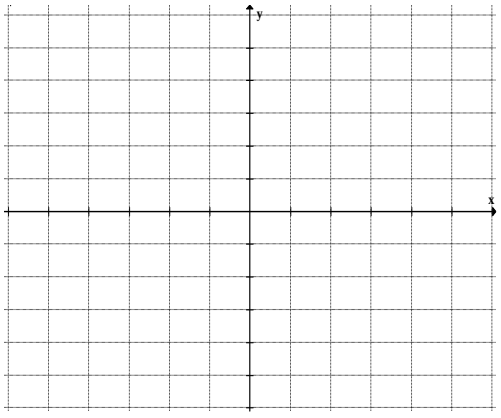
7. Find the equation of the ellipse having its center at $(-5,-1)$, a minor axis of length 10, and a vertex at $(-5,7)$.

8. Graph $(y-2)^2 = -8(x+1)$



10. Graph. Identify the vertex, focus, directrix and x-intercepts.

$$x^2 + 2y - 8x + 4 = 0$$



8.1 & 8.2 – Sequences and Series

Notation: $11, 14, 17, 20, 23, \cdots, 3n + 8,$

*Find the first four terms and the 100th term of each sequence:

1. $a_n = 8n - 5$

2. $a_n = (-1)^n(51n)$

With **recursive** sequences, rather than finding the n^{th} term by substituting in values for n, we find each term based on the terms before it.

*For each recursive sequence, list the first 5 terms.

3. $\begin{cases} a_1 = 87 \\ a_{n+1} = a_n + 12 \end{cases}$

4. $\begin{cases} a_1 = 4 \\ a_{n+1} = 3a_n - 7 \end{cases}$

5. $\begin{cases} a_1 = 7 \\ a_n = 10 - a_{n-1} \end{cases}$

6. $\begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} \end{cases}$

This is known as the Fibonacci sequence.

Summation (Sigma) Notation:

*Evaluate each of the following:

$\sum_{k=1}^{5}(2k+3)$

$\sum_{k=2}^{4}\left(\frac{k+1}{k+2}\right)$

Factorial Notation:

Definition: $n! = n(n-1)(n-2)\cdots 1$

Evaluate: $4!$

Evaluate: $6!$

By definition, $0!$ equals _____.

Simplify: $\frac{83!}{80!}$

Write the first four terms: $a_n = \frac{n!}{(n+2)!}$

Arithmetic Sequences

Arithmetic sequences are sequences in which a common _____ exists between terms.

Example: 2, 4, 6, 8, ... , 200, ...

Given $a_1 = 7$ and $d = 9$, find a_{48}.

Which term of 11, 16, 21, 26, ... is 1471?

For the arithmetic sequence with $a_5 = 47$ and $a_{11} = 113$, find a_1.

Add: $1 + 2 + 3 + \cdots + 998 + 999 + 1000$.

Given the sequence 2, 5, 8, 11, 14, ... find the following sums:

$S_1 =$

$S_2 =$

$S_3 =$

$S_4 =$

<u>Mini-Summary:</u>

$a_n = a_1 + (n-1)d$

$S_n = \frac{n}{2}(a_1 + a_n)$

Substituting the first (above) into the second give the following:

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$

Add: $103 + 99 + 95 + 91 + \cdots$ to 1437 terms.

For the given arithmetic sequence, the 82nd term, a_{82}, is equal to -370, and the 6th term, a_6, is equal to 10. Find the value of the 33rd term, a_{33}.

$$\sum_{k=5}^{971}(-4k + 117)$$

*Write each using sigma notation:

$87 + 92 + 97 + 102 + 107$

$172 + 168 + 164 + 160 + 156 + 152 + 148$

$\frac{1}{8}, \frac{2}{9}, \frac{3}{10}, \frac{4}{11}, \frac{5}{12},$

8.3 – Geometric Sequences and Series

Geometric Sequences and Series

Geometric sequences and series have a common _____ between terms.

Formulas: $a_n = a_1 r^{n-1}$

$$S_n = \frac{a_1 - a_1 r^n}{1-r}$$

Find a_8 if $a_1 = 4$ and $r = \frac{-1}{2}$.

Find a_7 if $a_1 = 64$ and $r = \frac{1}{4}$.

8.3 # 48 Find n for the geometric sequence having $a_1 = 1$, $a_n = -128$ and $r = -2$.

8.3 # 60 Find the common ratio r and the value of a_1 for the sequence having $a_5 = 6$ and $a_9 = 486$.

8.3 # 68 Find the sum S_8 when $a_1 = 12$ and $r = \frac{1}{2}$.

The sum of an infinite geometric series is given by $S_\infty = \frac{a_1}{1-r}$, provided $|r| < 1$.

Add: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

Note that if $r = 1$,
$2+2+2+2+2+\cdots$ has no infinite sum.

We say that this series _____.

Add: $\frac{3}{7} + \frac{1}{7} + \frac{1}{21} + \frac{1}{63} + \cdots$

Add: $60 - 30 + 15 - 7.5 + 3.75 - \cdots$

Review of sections 8.1 – 8.3

1. Given $a_1 = 62$, and $d = 7$, find a_{43} and S_{102}.

2. Find the first four terms of this recursive sequence: $\begin{cases} a_1 = 17 \\ a_n = 2a_{n-1} + 5 \end{cases}$

3. Write each of the following using sigma notation:

a. $131 + 122 + 113 + 104 + 95 + 86$

b. $1 + 8 + 27 + 64 + 125$

c. $40 + 20 + 10 + 5 + \frac{5}{2} + \frac{5}{4} + \frac{5}{8}$

d. $\frac{17}{53} + \frac{19}{50} + \frac{21}{47} + \frac{23}{44} + \frac{25}{41}$

4. Find both a recursive formula and a general formula for each sequence:

a. $4, 12, 36, 108, 324, \cdots$

b. $38, 45, 52, 59, 66, \cdots$

5. Find a_1 for the arithmetic sequence with $a_{43} = 218$ and $a_{51} = 507$.

6. Find the sum of the first one hundred even numbers.

8.4 – Proof by Induction

This topic is NOT currently covered in Aleks. It will be up to your teacher whether he/she will test on this section.

Steps:

1. Show that the statement is true for n=1.
2. Assume the statement is true for n=k.
3. Show that the statement is true for n=k+1.

Prove: $2 + 4 + 6 + 8 + \cdots + 2n = n(n + 1)$

Prove: $3 + 7 + 11 + \cdots + (4n - 1) = n(2n + 1)$

Prove: $4 + 12 + 20 + \cdots + (8n - 4) = 4n^2$

8.7 – The Binomial Theorem

Pascal's Triangle:

$(a + b)^0$=

$(a + b)^1$=

$(a + b)^2$=

$(a + b)^3$=

$(x + 3)^4$=

$(2x - 3y)^5$=

Combinations:

$$\binom{n}{r} = nCr = C(n,r) = \frac{n!}{r! \cdot (n-r)!}$$

Evaluate: $\binom{30}{2} =$

Calculator keystrokes:

Evaluate: $\binom{52}{5} =$

The Binomial Theorem:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

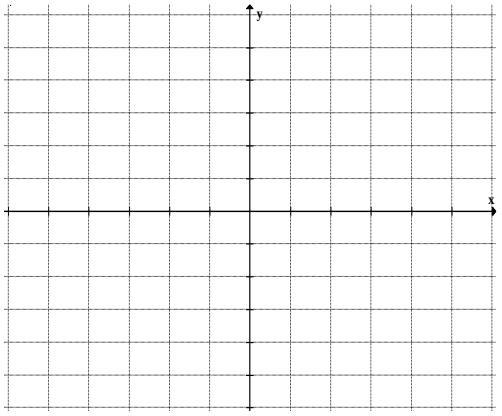
Find the 8th term of $(3x-2y)^{11}$=

Find the 6th term of $(2x+y)^{14}$=

Find the coefficient of the x^6y^9 term of $(2x-3y)^{15}$

Review of Chapters 7 & 8

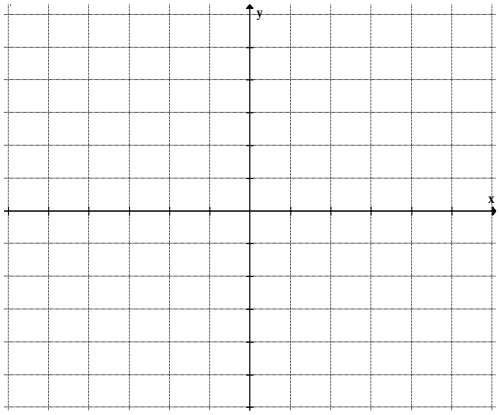
1. Graph. Label the foci. $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{1} = 1$



2. Find the center and foci:

$$4x^2 + 25y^2 - 16x - 50y - 59 = 0$$

3. Graph. $\frac{(y-2)^2}{4} - \frac{(x-1)^2}{16} = 1$



4. Name the foci and asymptotes above.

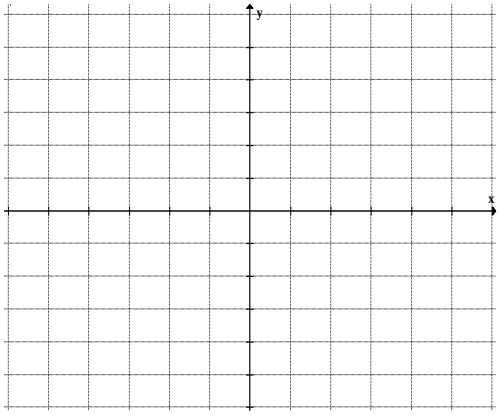
5. Find the center, vertices and foci:

$$4x^2 - 9y^2 - 40x + 36y + 28 = 0$$

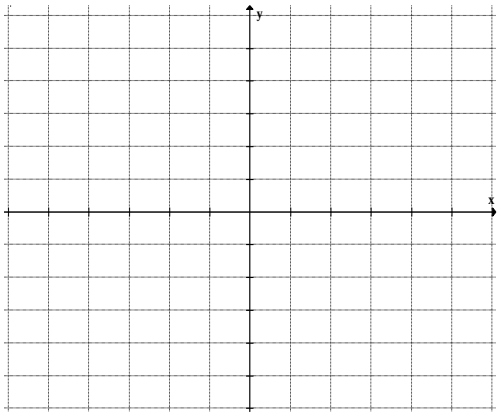
6. Find the equation of the ellipse with foci at $(5,1)$ and $(9,1)$ such that the minor axis is 4 units long.

7. Find the equation of the hyperbola with vertices at $(1,11)$ and $(1,-3)$ with foci at $(1,13)$ and $(1,-5)$.

8. Graph. $(y-2)^2 = 4(x+1)$



9. Graph. $(x-1)^2 = \frac{-1}{2}(y+3)$



10. Find the equation of the parabola with focus at $(-2,4)$ and directrix $x = 1$.

11. Find the vertex and focus of the parabola: $x^2 - 8x - 8y + 16 = 0$.

12. Expand: $(2x - 3y)^6$

13. Find the coefficient of the x^5y^{12} term of $(2x + y)^{17}$.

14. Prove by induction:

$$1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n - 1)}{2}$$

15. Find the sum: $40 + 30 + \frac{45}{2} + \frac{135}{8} + \cdots$

16. Find S_8 when $a_1 = 4$ and $r = \frac{1}{4}$.

17. Write each of the following using sigma notation:

a. $117 + 115 + 113 + 111 + 109$

b. $60 + 30 + 15 + 7.5 + 3.75$

18. Add to 117 terms: $18 + 23 + 28 + \cdots$

19. Given $a_{41} = 113$ and $a_{75} = 305$, find a_{211} for this arithmetic sequence. **(Aleks problem)**

20. For the arithmetic sequence with $a_5 = 32$ and $a_9 = 70$, find a_1.

21. Determine whether each is arithmetic or geometric. Then find a formula for the general term, a_n. **(Aleks problem)**

a. $5, 11, 17, 23, \ldots$

b. $4, 12, 36, 108, \ldots$

Final Exam Review

1. Solve: $x^2 - 12 \geq x$

2. Find the balance of an account after 5 years if \$3600 is invested at 4.5%, compounded monthly.

3. Multiply: $\begin{bmatrix} 2 & 3 & -4 \\ 5 & 1 & 1 \end{bmatrix}\begin{bmatrix} 6 & -8 \\ -1 & 3 \end{bmatrix}$

4. In 2004, a company had a net worth of \$43 million. In 2008, the same company had a net worth of \$62 million. Letting $t = 0$ represent the year 2004, find a function for the worth of the company as a function of time.

5. Given $A = \begin{bmatrix} -3 & 5 \\ -1 & 4 \end{bmatrix}$, find A^{-1}, if it exists.

6. Solve: $|2x + 7| - 3 \geq 5$

7. List the possible rational zeroes of the function $g(x) = 9x^4 + 5x^3 - 2x^2 + 4$

8. Name all asymptotes and intercepts of $f(x) = \frac{x-3}{2x+1}$.

9. For the rational function described in exercise 8, finish the following statements:

$As\ x \to -\infty, y \to$_____

$As\ x \to +\infty, y \to$_____

10. How long would it take \$1500 to double when invested at 2.2%, compounded daily?

11. Find the vertices of the polygon formed:

$$\begin{cases} x \geq 2 \\ y \geq 1 \\ y \leq -x + 11 \\ y \geq x + 3 \end{cases}$$

12. Find S_{143} for the sequence $\frac{3}{5}, 1, \frac{7}{5}, \frac{9}{5}, \ldots$

13. Find the domain of each:

a. $f(x) = \sqrt{3x - 5}$

b. $g(x) = \frac{x+2}{3x-5}$

c. $h(x) = log_4(3x - 5)$

d. $f(x) = \sqrt{x^2 - 25}$

14. Find S_5 for the geometric sequence with $a_1 = 6$ and $r = -2$.

15. Find the sum, if possible:

$$4 + 4(-3) + 4(-3)^2 + 4(-3)^3 + \cdots$$

16. Find the focus and directrix of

$$(y + 3)^2 = -12(x + 2)$$

17. The length of a rectangle is 4 cm more than its width. Find a function which represents the rectangle's area in terms of its width.

18. Find the foci:

$$x^2 - 4y^2 - 12x - 8y + 16 = 0.$$

19. Sketch the graph of

$$y = (x+3)^2(x-4)(x-1)^4$$

20. What does Descartes' Rule of Signs tell us about the possible number of positive and negative real zeroes of p(x)?

$$p(x) = 2x^7 + x^6 + x^5 - x^4 - x^3 + x^2 - x + 2$$

21. Solve: $2^{x+3} = 17$