Math 226-DLA Limits by Rationalization or Multiplication by the Conjugate

OBJECTIVE: Students will learn to find limits analytically by rationalization or multiplication by the conjugate.

When to Use Rationalization or Multiplication by the Conjugate

If the function you are taking the limit of contains a radical function or trigonometric function that has two terms that are adding or subtracting, try multiplying by the conjugate.

Multiplying by the Conjugate Examples:

EXAMPLE: $\lim_{x\to 10} \frac{\sqrt{x-6}-2}{x-10}$

SINCE DIRECT SUBSTITUTION GIVES US A ZERO IN THE DENOMINATOR, WE MUST THINK OF OTHER TECHNIQUES WE CAN USE TO EVALUATE THE LIMIT. SINCE WE SEE A RADICAL EXPRESSION WITH TWO TERMS THAT ARE SUBTRACTING IN THE NUMERATOR, LET'S TRY MULTIPLYING BY THE CONJUGATE OF THE NUMERATOR.

$$\lim_{x\to 10} \frac{\sqrt{x-6}-2}{x-10} = \lim_{x\to 10}\left(\frac{\sqrt{x-6}-2}{x-10}\cdot\frac{\sqrt{x-6}+2}{\sqrt{x-6}+2}\right)$$

TIP: DON'T DISTRIBUTE IN THE DENOMINATOR.

$$= \lim_{x\to 10} \frac{x-6-4}{(x-10)(\sqrt{x-6}+2)}$$

$$= \lim_{x\to 10} \frac{\cancel{x-10}}{\cancel{(x-10)}(\sqrt{x-6}+2)}$$

$$= \lim_{x\to 10} \frac{1}{\sqrt{x-6}+2}$$

$$= \frac{1}{\sqrt{10-6}+2}$$

$$= \frac{1}{\sqrt{4}+2}$$

$$= \frac{1}{4}$$

EXAMPLE: $\lim_{\theta \to 0} \frac{3\sin^2\theta}{1-\cos\theta}$

SINCE DIRECT SUBSTITUTION GIVES US A ZERO IN THE DENOMINATOR, WE MUST THINK OF OTHER TECHNIQUES WE CAN USE TO EVALUATE THE LIMIT. SINCE WE SEE A TRIGONOMETRIC EXPRESSION WITH TWO TERMS THAT ARE SUBTRACTING IN THE DENOMINATOR, LET'S TRY MULTIPLYING BY THE CONJUGATE OF THE DENOMINATOR.

$$\lim_{\theta \to 0} \frac{3\sin^2\theta}{1-\cos\theta} = \lim_{\theta \to 0} \left(\frac{3\sin^2\theta}{1-\cos\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta}\right)$$

$$= \lim_{\theta \to 0} \frac{(3\sin^2\theta)(1+\cos\theta)}{1-\cos^2\theta}$$

PYTHAGOREAN IDENTITY: $\cos^2\theta + \sin^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$$= \lim_{\theta \to 0} \frac{(3\sin^2\theta)(1+\cos\theta)}{\sin^2\theta}$$

$$= \lim_{\theta \to 0} 3(1+\cos\theta)$$

$$= 3[1+\cos(0)]$$

$$= 3(1+1)$$

$$= 6$$

Try these on your own.

1. $\lim_{x \to 0} \frac{3-\sqrt{x+9}}{x}$
2. $\lim_{x \to 5} \frac{\sqrt{3x-1}-\sqrt{2x+4}}{x-5}$
3. $\lim_{x \to 0} \frac{\cos x - 1}{\sin x}$
4. $\lim_{x \to 0} \frac{\sqrt{x+10}-\sqrt{10}}{x}$
5. Why is it "legal" in the second example the multiply the function you are finding the limit of by $\frac{1+\cos\theta}{1+\cos\theta}$?
6. What are $1+\cos\theta$ and $1-\cos\theta$ called?
7. It will be helpful on your homework or an exam to quickly be able to recognize when you need to multiply by the conjugate. What are some characteristics of a function that would make you think to use this technique?